Diffraction and Propagation Models Based on the Parabolic Wave Equation.

A.V.Popov
Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation
IZMIRAN, Troitsk, Moscow region, 142190, Russia
E-mail: popov@izmiran.rssi.ru

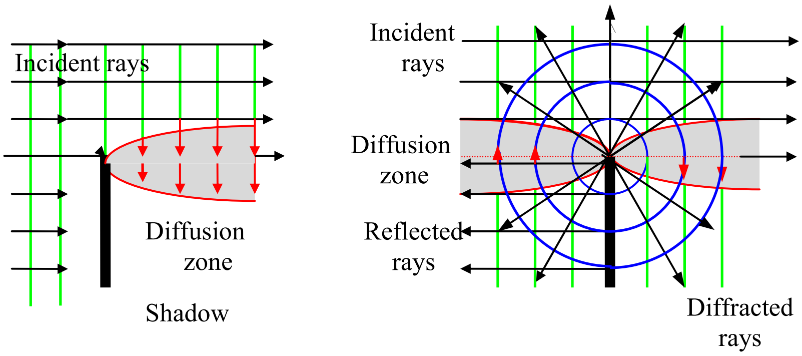
Computational aspects of the parabolic equation method are discussed, including a short historical note. Derivation of vectorial PE for nonuniform 3D environments is given in more detail. New applications to radio wave propagation, X-ray optics and microwave holography are mentioned.

1. Historical Notes

The first paper on the parabolic wave equation was published by M.A.Leontovich in 1944. It called wider attention after his joint publication with V.A.Fock [1] related to the important problem of radio propagation over the realistic earth surface. The reduction of a rigorous boundary value diffraction problem to a simple initial problem for a Schroedinger type "parabolic" equation

$$2ik\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (1)$$

gave a variety of new analytical solutions [2] and opened an exciting era of euristic study of wave phenomena which can be compared with the golden age of Young and Fresnel's works.



a). Leontovich's model b). Malyuzhinets' model

Fig.1. Diffraction by half plane

G.D.Malyuzhinets was the first one who realized the general character of the parabolic approximation and its enormous computational potential. In his report to an electrotechnical conference in Gorki (1946) a bold generalization of the PE method was proposed, combining it with the well-known ray coordinates of geometric optics. This idea, elaborated during the next decade, has resulted in a new diffraction theory [3] consonant to Young's pioneer concepts and alternative to the famous Keller's geometric theory of diffraction. Malyuzhinets' theory, named by its author "the transversal diffusion method", explains diffraction phenomena as lateral transfer of the complex wave amplitude along curvilinear wave fronts obeying the generalized Fermat principle. This model includes the concept of diffracted rays and clarifies the mechanisms of the energy penetration into the shadow zones both in the case of the obstacle with sharp edges (Fig.1) and the smooth convex body (Fig.2)

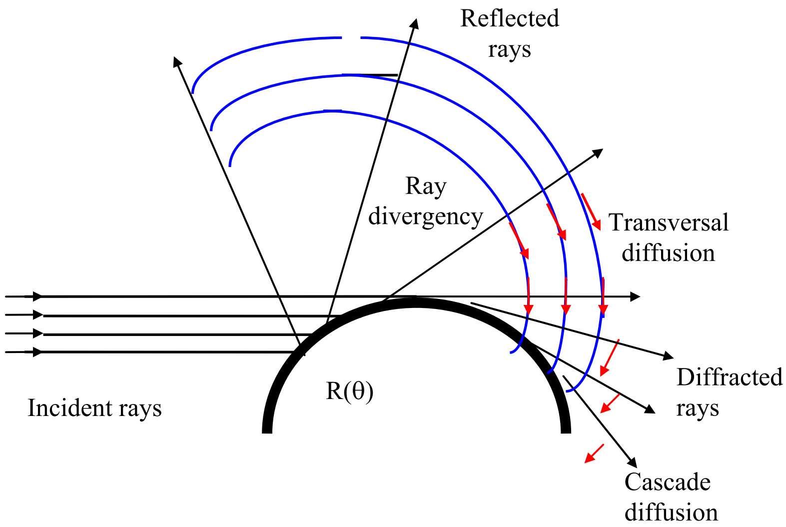


Fig.2. Malyuzhinets' model of diffraction by convex body.

Mathematically, Malyuzhinets' model is based on the generalized PE written in curvilinear ray coordinates *(ξ,η,ζ)* describing partial wave fields (incident, reflected and diffracted)

$$ik\left[2\frac{\partial u}{\partial \xi}+\frac{\partial \ln\left(h_\eta h_\zeta\right)}{\partial \xi}u\right]+\frac{1}{h_\eta h_\zeta}\left[\frac{\partial}{\partial \eta}\left(\frac{h_\zeta}{h_\eta}\frac{\partial u}{\partial \eta}\right)+\frac{\partial}{\partial \zeta}\left(\frac{h_\eta}{h_\zeta}\frac{\partial u}{\partial \zeta}\right)\right]=0 \qquad (2)$$

Following Malyuzhinets, the complete diffraction problem should be posed on a multi-sheet Riemann manifold, each sheet corresponding to one partial ray family and being matched to another one along the obstacle surface or the shadow boundary.

This general scheme provides full qualitative description of short-wave diffraction phenomena and accurate quantitative results for a number of model examples admitting exact analytical solution. It agrees with the original Leontovich-Fock approach inside the effective diffusion zones where usual PE (1) can be applied to matching two or more partial wave fields. Outside the critical zones, diffusion stops and Malyuzhinets PE reduces to ordinary geometric optics or diffractive ray optics of GTD:

$$E = u\exp(ik\xi), \qquad u = \frac{v(\eta,\zeta)}{\sqrt{h_\eta h_\zeta}} \tag{2a}$$

The transversal diffusion principle, along with J.B.Keller's concept of diffracted rays, Leontovich's parabolic approximation and Fock's asymptotic techniques, formed a firm basis for future development of physical diffraction theory. Mathematical refinement and further development of the parabolic equation method has been achieved, among others, in the well-known works of V.M.Babich, V.S.Buldyrev and their colleagues [4].

G.D.Malyuzhinets was one of the pioneers of introducing computers and numerical methods into diffraction theory. He anticipated the great practical need in wave calculations for radar technology and underwater acoustics and was not satisfied with the series of new analytical solutions obtained by means of the parabolic equation method. In his lectures, he used to point out the computational advantages offered by the parabolic approximation: initial value problem instead of complete boundary value one, absolute numerical stability, slowly varying wave attenuation function instead of rapidly oscillating primary wave field.

First attempts to solve Leontovich's PWE by means of finite difference method were made in Malyuzhinets' research group around 1960. He suggested using the Crank-Nicholson FD scheme retaining energy conservation law and asymptotic stability of the differential PE. By 1964, Leontovich's PE (1) was applied to numerical simulation of underwater acoustic propagation, and the first numerical solutions of the general equation of transversal diffusion (2) were constructed by means of finite differences. Malyuzhinets' ambitious research program announced in [5] included constructing numerical algorithms for a wide class of diffraction and propagation problems (objects of arbitrary shape, nonuniform media etc.) using both original Leontovich equation (1) and general Malyuzhinets' PE (2). Of course, full success could not be reached as powerful computers and visualization tools were not available those times. Nevertheless, many of Malyuzhinets' ideas were implemented and developed by his former students and colleagues (V. Zavadsky [6], E. Polyansky [7], Yu. Cherkashin [8], A. Popov [9-10] and others). I don't know whether F.Tappert was aware of Malyuzhinets' group activity in underwater acoustics, but his famous lecture on the parabolic approximation [11] followed Malyuzhinets' ideas and contained necessary references. This paper and brilliant J.Claerbout's book on seismic data processing [12] made PE popular in the Western diffraction community and gave a strong impulse to the development of computational acoustics. Most of further applications of the PE to radio propagation combine Fock's analytical approach with Malyuzhinets' idea of straightforward numerical integration. Due to many improvements and modifications proposed during the last two decades (long-range and wide angle versions, artificial transparent boundaries, realistic tropospheric and irregular terrain models) the PE method has become an extremely powerful and versatile computational tool in modern electromagnetic theory - see recently published excellent book by M.Levy [13].

2. Ways of Further Development

Malyuzhinets' transversal diffusion method has been developed for scalar acoustical wave equation, and actually only two-dimensional problems were treated. Leontovich and Fock started from 3D vectorial Maxwell equations, however, due to symmetry, the problem was reduced to a scalar 2D parabolic equation. Nowadays, many radar and propagation applications require full 3D vector analysis. Hence, a number of analytical and numerical problems arise:

- consistent and reliable derivation of 3D vector parabolic equations
- 3D generalization of transparent boundary conditions
- completing and numerical implementation of Malyuzhinets' transversal diffusion model
- development of simplified diffraction and propagation models aimed at specific applications.

Derivation of vector parabolic equations.

In free space, each Cartesian component of the EM field satisfies scalar Helmholtz equation. Therefore, 6 parabolic equations can be written down for the corresponding slowly varying wave amplitudes. However, they are not independent, and some components can be eliminated from the boundary value problem formulation. For nonuniform media, reduction of Maxwell's equation to a vector PE is not trivial and depends on the environment model. Two examples have been studied by Baranov and Popov [14-15] using 2-scale asymptotic analysis:

a). EM wave propagation in a nonuniform dielectric layer [14]. Dielectric permittivity ε and layer thickness b are assumed to be slowly varying functions of horizontal variables X,Y: $\varepsilon = \varepsilon(X/L, Y/L)$, $b = Bh(X/L, Y/L)$ with the characteristic scale $L >> B$. On the other hand, the reference thickness B is large compared with the wavelength: $B >> \lambda$. We introduce small parameter

$$\nu \sim \frac{\lambda}{B} \sim \frac{B}{L} \tag{3}$$

and seek an asymptotic solution to the Maxwell equations in the following form:

$$\vec{\Pi}(x,y,z) = \vec{U}(x,y,z)\exp\left[i\frac{\sigma}{\nu^2}\Phi(x,y) \right]$$

Here, $\vec{\Pi} = \left(E_x, E_y, E_z, H_x, H_y, H_z\right)^T$ is the full EM field vector, $\sigma = B^2/\lambda L$ is dimensionless Fresnel number, phase Φ and 6-component wave amplitude $\vec{U}$ being functions of properly scaled Cartesian coordinates: $x = X/L,\ y = Y/L,\ z = Z/B$.

Substitution into Maxwell's equation and eliminating secular terms in the asymptotic expansion

$$\vec{U}(x,y,z) = \vec{U}_0 + \nu\vec{U}_1 + \nu^2\vec{U}_2 + \ldots \tag{4}$$

yield the eikonal equation

$$(\nabla\Phi)^2 = \varepsilon(x,y) \tag{5}$$

and a pair of parabolic equations

$$\begin{aligned} & i\sigma(2\nabla\Phi\nabla u + u\Delta\Phi) + \frac{\partial^2 u}{\partial z^2} = 0 \\ & i\sigma\left[2\nabla\Phi\nabla v + \left(\Delta\Phi + \frac{\nabla\varepsilon\nabla\Phi}{\varepsilon}\right)v\right] + \frac{\partial^2 v}{\partial z^2} = 0 \end{aligned} \tag{6}$$

governing scalar functions *u(x,y,z), v(x,y,z)* being amplitudes of two polarization vectors

$$\vec{a} = \left(0,0,1,\Phi_y,-\Phi_x,0\right)^T, \quad \vec{b} = \left(-\Phi_y,\Phi_x,0,0,0,\varepsilon\right)^T \tag{7}$$

in the zero-order approximation

$$\vec{U}_0(x,y,z) = u(x,y,z)\vec{a} + v(x,y,z)\vec{b}\,. \tag{8}$$

So, EM waves in a nonuniform dielectric layer propagate along horizontal rays determined by Eq.(5). Two polarization modes *u(x,y,z)* and *v(x,y,z)*, within precision of order $0(\nu)$, propagate independently, governed by the laws of ray divergence in the horizontal plane *(x,y)* and transversal diffusion in vertical direction *z*.

b). Our second example relates to UHF radio propagation in tunnels [15]. Mathematically, the problem consists in constructing propagation modes in a smoothly curved waveguide of arbitrary cross section. It is too complicated to be treated in the exact form. On the other hand, the low-order modes with small grazing angles, having minimum attenuation, can be described by a 3D vector PE. Moreover, typically the tunnel axis curvature radius $R(s) \sim L$ is so large that the Fresnel number B^2/L appears to be a small parameter $0(\nu)$ where $\nu \sim \lambda/B$. Instead we introduce Fock's dimensionless number $\sigma = B^{3/2}/\lambda R^{1/2} = 0(1)$, where *B* is a reference cross section diameter. Using the modified Ansatz written in scaled longitudinal and transverse variables $\xi = s/L, \quad \eta = q/b, \quad \zeta = z/b$

$$\vec{\Pi}(\xi,\eta,\zeta) = \vec{U}(\xi,\eta,\zeta)\exp\left[i\frac{\sigma}{\nu^3}\Phi(\xi)\right], \tag{9}$$

the eikonal $\Phi(\xi) = \xi + \nu^2\Phi_1(\xi)$ and the vector amplitude

$$\vec{U}(\xi,\eta,\zeta) = \vec{U}_0 + \nu\vec{U}_1 + \nu^2\vec{U}_2 + \ldots \tag{10}$$

being functions to be found, we get a recurrent set of linear algebraic equations for the coefficients $\vec{U}_n(\xi,\eta,\zeta)$:

$$\begin{cases} \hat{S}\vec{U}_0 = 0, \\ \hat{S}\vec{U}_n = F\left[\vec{U}_{n-1}, \vec{U}_{n-2}, \vec{U}_{n-3}\right] \end{cases}. \tag{11}$$

Here, $\hat{S}$ is a constant, twice degenerating matrix. Therefore, general solution of the zero-order equation has the form

$$\vec{U}_0(\xi,\eta,\zeta) = u_0(\xi,\eta,\zeta)\vec{a} + v_0(\xi,\eta,\zeta)\vec{b} \tag{12}$$

where $\vec{a} = (0,1,0,0,0,1)^T$, $\vec{b} = (0,0,1,0,-1,0)^T$ are constant annuling eigenvectors, and u_0, v_0 are arbitrary functions of the scaled variables (ξ,η,ζ). These functions can be found from the solvability conditions of the higher order equations. They appear to be proportional to an eigenfunction of the transversal Schroedinger operator

$$\begin{aligned} &\frac{\partial^2 w}{\partial\eta^2} + \frac{\partial^2 w}{\partial\zeta^2} - 2\sigma^2\chi(\xi)\eta w = 2\sigma^2\gamma w, \\ &w|_\Gamma = 0. \end{aligned} \tag{13}$$

Here, $\chi(\xi) = L/R(s)$ is normalized curvature of the waveguide axis in *(X,Y)* plane, and the spectral parameter $\gamma = \Phi_1'(\xi)$ determines dispersion of the mode phase velocity. Both eigenvalues $\gamma_{mn}(\xi)$ and eigenfunctions $w_{mn}(\xi,\eta,\zeta)$ depend on the cross section shape Γ and the waveguide axis curvature $\chi(\xi)$. The amplitude coefficients $A(\xi)$, $B(\xi)$ in the reduced EM field representation

$$\vec{W}_0 \equiv \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} w_{mn}(\xi,\eta,\zeta) \tag{14}$$

can be found in the process of constructing the subsequent corrections to the zero order solution (12). The problem reduces to the solvability condition of the inhomogeneous Schroedinger equation

$$\Delta_\perp\vec{W}_1 - 2\sigma^2(\gamma + \chi\eta)\vec{W}_1 = -2i\sigma\frac{\partial\vec{W}_0}{\partial\xi} \tag{15}$$

with an inhomogeneous boundary condition

$$\vec{W}_1\big|_{\Gamma} = \hat{G}\left[\vec{W}_0\right] \tag{16}$$

depending on the wall impedance $Z(\xi,\eta,\zeta)$. It results in a linear set of ordinary differential equations for the coefficients $A(\xi), \quad B(\xi)$:

$$\begin{aligned} A' &= P_{11}A + P_{12}B \\ B' &= P_{12}A + P_{22}B \end{aligned} \tag{17}$$

Their solution determines attenuation and depolarization of the two-component vector modes of arbitrary shaped slowly nonuniform waveguides.

This asymptotic analysis allows one to avoid numerical integration of 3D differential equations due to approximate separation of variables. Still, an equivalent form of solution, which can be derived from the above analysis, may be preferable. Combining Eqs.(13), (15) shows that two-component vector function

$$\vec{W}(\xi,\eta,\zeta) = (\vec{W}_0 + \gamma\vec{W}_1)e^{i\frac{\sigma}{\nu}\Phi_1(\xi)} \tag{18}$$

satisfies the following vector parabolic equation

$$2i\nu\sigma\frac{\partial\vec{W}}{\partial\xi} + \Delta_{\perp}\vec{W} - 2\sigma^2\chi(\xi)\eta\vec{W} = 0 \tag{19}$$

adequately describing an arbitrary superposition of low-qrder modes of the oversized nonuniform waveguide. Its numerical step-by-step integration serves as an efficient method of field calculations in tunnel environments [16].

Artificial transparent boundaries.

Most of diffraction and propagation problems are posed in free space or in a regular semi-infinite background medium which creates specific difficulties when using straightforward numerical methods. It is always desirable, if not necessary, to confine the computational domain to a minimum volume containing scattering objects and the observation point. One of important advantages of the parabolic, equation is the possibility to perform such reduction without additional approximations. An exact method of such equivalent confinement has become widely used after simultaneously published papers [17,18]. It is based on the PE analytical solution in the regular semi-infinite subdomains to be cut off. On this way, a linear singular operator arises relating the normal derivative of the solution to be found with its distribution over the artificially introduced finite boundary $z = a$:.

$$\frac{\partial u}{\partial z}(x,a) = -e^{-i\pi/4}\sqrt{\frac{2k}{\pi}}\int_0^x \frac{\partial u}{\partial x}(\xi,a)\frac{d\xi}{\sqrt{x-\xi}} \qquad (20)$$

This nonlocal boundary operator, being a generalization of the absorbing surface impedance, simulates perfectly transparent boundary and produces no artifacts in numerical implementation. This method has been successfully used in a number of radio wave, acoustical and X-ray applications. A review of author's work on transparent boundary conditions has been given in [19]. Recently, TBC for 3D PE calcutions in cylindrical and rectangular domains have been derived. Their numerical implementation and vector generalization for full-wave EM analysis are the nearest goals.

Transversal diffusion in ray coordinates.

Malyuzhinets' idea of global field description via PE written in ray coordinates seems to be very promising for high precision calculations in a wide angular region. Of course, Eq.(2) looks complicated compared with the original Leontovich PE (1), and the procedure of superimposing many partial wave fields may be rather cumbersome. However, the difficulties may appear not to be so heavy. As 2D examples show, Malyuzhinets equation (2) in free space can be exactly reduced to the simplest form of Eq.(1) by change of variables. For convex bodies, it gives an efficient method of scattered field calculation. A basic problem of mapping from the ray coordinates to physical space remains in the case of focusing by concave reflectors. 3D and vector generalizations of Malyuzhinets' transversal diffusion method not only are extremely interesting but may have essential practical impact.

Simplified diffraction models.

Original Leontovich PE (1), along with its wide-angle and long-range modifications, remains to be one of the most efficient computational tools in underwater acoustics and radio propagation theory. Among new applications, numerical simulation of X-ray diffractive optics developed in [20-21] is to be mentioned. The parabolic approximation proved to be highly adequate due to low refractivity of all available materials in this spectral region and relatively small numerical aperture of X-ray optical schemes. Finite difference solution of the nonuniform PE

$$2ik\frac{\partial u}{\partial z} + \Delta_{\perp}u + k^2(n^2-1)u = 0 \qquad (21)$$

with complex refraction index $n = 1-\delta + i\beta, \quad \delta,\beta << 1$ allows one to estimate imaging performance of realistic Fresnel zone plates (spatial resolution, diffraction efficiency, chromatic aberration etc.) as function of their geometry and material properties. Gain in numerical efficiency is so high that fine Fresnel lenses with hundreds of zones can be simulated with high precision on personal computers. Many diffraction effects can be investigated analytically using PE as it allows for a wider class of exact solutions compared with the exact wave equation (eg.: weak dielectric wedge or pinhole in a thick plane screen being of interest for contact EUV lithography).

In its original formulation, the parabolic equation method ignores wide angle scattering and backward reflection. Nevertheless, combining standard Leontovich PE(1) with GO and GTD to provide adequate BC and using transparent boundary conditions for the domain truncation one can construct an efficient practical algorithm for radar backscatter simulation (Fig. 3).

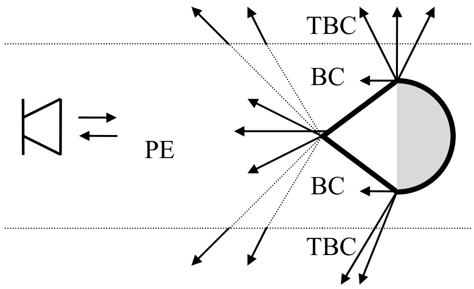


Fig.3 Paraxial backscatter model

Recently this scheme has been extended to the wide-angle radar scattering by using a number of different paraxial axes [13].

Inverse holographic migration.

The near-field holographic subsurface radar being developed at IZMIRAN uses the complex values of the scattered field amplitude measured by a rectangle receiving antenna array in order to reconstruct the position and properties of the scatterer. An evident practical solution is backward wave field migration from the antenna plane into the half space embedding the scattering object. The main field maxima are supposed to indicate the discrete scatterer positions. In the harrow-angle approximation, such field migration can be performed by means of the Fresnel-Kirchhoff integral formula or, equivalently, by numerical integration of the Leontovich PE (1) (the latter approach admits a straightforward generalization to the case of a nonuniform background medium). An alternative approach consists in correlating the measured wave field with a three-parameter family of slanting Gaussian beams supported by the receiving aperture $|x| < A, |y| < A$ and focused onto a certain "plane of best vision" $z = \ell$. Maxima of the resulting angular spectrum, depending on the range variable ℓ as a parameter, give an immediate localization of the virtual scatterers [22].

References

1. M.A.Leontovich, V.A.Fock. Solution of the problem of EM wave propagation along the earth surface by the parabolic equation method, *J.Phys.,* Moscow, 10(1), 13, 1946.

2. V.A.Fock. Electromagnetic. Diffraction and Propagation Problems, *Internat Series of Monographs on Electromagnetic Waves,* 1, Pergamon Press Oxford, 1965.
3. G.D.Malyuzhinets. Progress in understanding diffraction phenomena (in Russian), *Sov. Phys. Usp*., Engl. Transl., 69(2), 321-334, 1959.
4. V.M.Babich, V.S.Buldyrev. Short-Wavelength Diffraction Theory: Asymptotic Methods, 4 , *Springer Series on Wave Phenomena*, Springer-Verlag, Berlin, 1991.
5. G.D.Malyuzhinets, A.V.Popov, Yu.N.Cherkashin. On the development of a numerical method in diffraction theory (in Russian), in *Proceedings of the 3rd Symposium on Diffraction,* 176-178, Nauka, Moscow, 1964.
6. V.Yu.Zavadsky. Wave *Process Modeling* (in Russian), Nauka, Moscow, 1991.
7. E.A.Poliansky. On the relation between solutions of Helmholtz and Schroedinger type equations (in Russian), *Comp. Math. and Math. Phys*., 2(1), 241-249, 1972.
8. Yu.N.Cherkashin. Calculation of wave fields in smoothly nonuniform media by means of the parabolic equation method (in Russian), in: *Decameter Radiowave Propagation*, IZMIRAN, Moscow, 5-18, 1980.
9. A.V.Popov. Numerical solution of the wedge diffraction problem by the transversal diffusion method, *Sov. Phys. Acoust*., 15(2), 226-233, 1969.
10. A.V.Popov, S.A.Hoziosky. On a generalized parabolic equation of diffraction theory (in Russian), *J. Comp. Math. Phys*., 17(2), 527-533, 1977.
11. F.D.Tappert. The parabolic approximation method, in *Lecture Notes in Physics, Note 70, Wave Propagation and Underwater Acoustics*, 224-287, Springer Verlag, New York, 1977.
12. J.F.Claerbout. Fundamentals of Geophysical Data Processing with Application to Petroleum Prospect, *McGraw-Hill*, New York, 1976.
13. M.Levy. Parabolic Equation Methods for Electromagnetic Wave Propagation, *IEE Electromagnetic Waves Series,* 45 , London, 2000.
14. V.A.Baranov, A.V.Popov. Generalization of the parabolic equation for EM waves in a dielectric layer of nonuniform thickness, *Wave Motion*, 17, 337-347, 1993.
15. V.A.Baranov, A.V.Popov. Adiabatic modes of curved EM waveguides of arbitrary cross section, in: *13thj Annual Review of Progress in Applied Computational Electromagnetics*, 1036-1044, Monterey, CA, 1997.
16. A.V.Popov, V.A.Vinogradov, N.Y.Zhu, F.M.Landstorfer. 3D parabolic equation model of EM wave propagation in tunnels, *Electronics Letters*, 35, No 11, 880-882, 1999.
17. V.A.Baskakov, A.V.Popov. Implementation of transparent boundaries for numerical solution of the Schroedinger equation, *Wave Motion*, 14(1), 123-128, 1991.
18. S.V.Marcus. A generalized impedance method for application of the parabolic approximation to underwater acoustics, *J. Acoust. Soc. Am*., 89, 391-398, 1991.
19. A.V.Popov. Accurate modeling of transparent boundaries in quasi-optics, *Radio Science*, 31, N^0 6, 1781-1790, 1996.
20. Yu.Kopylov, A.V.Popov, A.V.Vinogradov. Application of the parabolic wave equation to X-ray diffraction optics, *Opt. Commun*., 118, 619-636, 1995.
21. A.V.Vinogradov, A.V.Popov, Yu.V.Kopylov, A.N.Kurokhtin. Numerical Simulation of X-ray Diffractive Optics, *Moscow X-ray Optics Group*, A&B Publishing House, 1999.

22. A.V.Popov, V.V.Kopeikin, V.A.Vinogradov. Holographic subsurface radar: numerical simulation, *Proc. 8th Internat. Conf. On Ground Penetrating Radar*, Gold Coast, Australia, 288-299, 2000.