METHOD OF ANALYTICAL REGULARIZATION BASED ON THE STATIC PART INVERSION IN WAVE SCATTERING BY IMPERFECT THIN SCREENS

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Summary. *The paper is focussed on the development of the Method of Analytical Regularization (MAR) in electromagnetic wave scattering and absorption by imperfect thin screens.*

1. IMPERFECT BOUNDARY CONDITIONS

As known (see [1-6]), if the thickness of imperfect scatterer is small compared to the free-space wavelength, the wave scattering problem can be simplified to exclude the internal field from consideration. This is done by assuming the scatterer thickness to be zero but at the same time introducing specific boundary conditions modified with respect to the perfectly electric conducting (PEC) boundary conditions. Here, certain *effective* parameters appear, accumulating the values of the thickness and material constants of the scatterer. These parameters couple together the limit values of the tangential field components $\vec{E}_T^{\pm}$ and $\vec{H}_T^{\pm}$ on two sides of the scatterer. The most general form of the mentioned conditions is (see [4]):

$$\frac{1}{2}[\vec{E}_T^+ + \vec{E}_T^-] = Z_0 R \vec{n} \times [\vec{H}_T^+ - \vec{H}_T^-] + W[\vec{E}_T^+ - \vec{E}_T^-], \tag{1}$$

$$\frac{1}{2}[\vec{H}_T^+ + \vec{H}_T^-] = -Z_0^{-1} S \vec{n} \times [\vec{E}_T^+ - \vec{E}_T^-] - W[\vec{H}_T^+ - \vec{H}_T^-], \tag{2}$$

Basically, there are three particular forms of (1), (2). The first one appears in the case of a thinner-than-skindepth metal scatterer of finite conductivity: $R \neq 0$, $S = \infty$, $W = 0$; the second one is derived for a thin magneto-dielectric scatterer (material): $R \neq 0$, $S \neq \infty$, $W = 0$; and the third one for a PEC scatterer covered with a thin material layer: $R \neq 0$, $S \neq \infty$, $W \neq 0$. Zero-thickness scatterers are commonly called *screens* and can be flat or curved. Of the three mentioned types of screens, the first two (resistive and material) are partially transparent while the third one (impedance) is non-transparent.

So, any type of thin scatterers can be simulated by using at most three parameters. There exists an ambigous terminology about them. We shall use the one proposed in [4] and call them *resistivities*: electric R, magnetic S, and so-called cross-resistivity W. For example, in the case of a thinner-than-skindepth metal screen: $R = (Z_0 h \sigma)^{-1}$, where h is the thikness, σ is the conductivity, Z_0 is the free-space impedance [2,4,5]. In the case of a thin *single-layer* material screen [1,3,4] having normalized parameters ϵ_r and μ_r, one obtains that $R = (i/2) Z \cot\left[(1/2)\epsilon_r^{1/2}\mu_r^{1/2} k_0 h\right]$, $S = R/Z^2$, and $W = 0$, where h is the thickness, $Z = (\mu_r/\epsilon_r)^{1/2}$, and k_0 is the free-space wavenumber. Note that for a multi-layer sandwich-like material slab, $W \neq 0$ [4].

It is interesting to note that resistive and material conditions were proposed empirically extensively used for quite a long time (e.g., see [7-10]), before being fully grounded in [2] and [3], respectively. Mathematically rigorous derivation of the mentioned expressions for the resistivities was also done in [2,3] confirming more relaxed derivations of [1,4-6]. For the sake of completeness, it should be mentioned that material screens have been also simulated by introducing the so-called "higher-order" imperfect boundary conditions [5]. The latter involve not only the limit values of tangential fields but also their normal derivatives. For example, condition (2) is modified to take the following form (see [11]):

$$\frac{1}{2}[\vec{H}_T^+ + \vec{H}_T^-] = -\frac{1}{Z_0}\vec{n} \times \left(S - \frac{\partial}{k\partial n}\right)[\vec{E}_T^+ - \vec{E}_T^-]$$

In this paper, we shall base our considerations on the "zero-order" conditions (1), (2) although an extension of MAR to the conditions of [11] is also possible. Note that conditions of the order higher than 1 are ambigous from the viewpoint of solution uniqueness [12].

In the case of impenetrable thin imperfect screen, the resistivities are not independent and should satisfy the relationship derived in [4]: $4(RS + W^2) = 1$. Therefore, the pair of the *impedance* boundary conditions (1) and (2) can be also formulated in terms of two other effective parameters: *surface impedances* Z^+ and Z^-, i.e., as the two-side *Leontovich boundary conditions* [4-6]:

$$\vec{E}_T^\pm \mp Z_0 Z^\pm \vec{n} \times \vec{H}_T^\pm = 0, \qquad (3)$$

which are equivalent to (1) and (2) provided that

$$R = \frac{Z^+Z^-}{Z^+ + Z^-}, \quad S = \frac{1}{Z^+ + Z^-} \quad W = \frac{1}{2}\frac{Z^+ - Z^-}{Z^+ + Z^-}, \quad Z^+ + Z^- \neq 0 \qquad (4)$$

For example, a thin PEC screen coated with different layers of magneto-dielectrics of parameters $h^\pm$, $\epsilon_r^\pm$ and $\mu_r^\pm$, has the values of the surface impedances given by [4-6]: $Z^\pm = -iZ\tan[(\epsilon_r^\pm \mu_r^\pm)^{1/2} k_0 h^\pm]$. It is important to note that if any material parameter of an imperfect screen is not real but complex-valued, then the resistivities obtain non-zero real parts, which are responsible for the dissipation losses. Therefore, by modifying the boundary conditions, one can study not only the wave scattering but also the wave absorption. Besides, if any of the quantities ϵ_r, μ_r or h varies along the screen surface, then the resistivities R, S, W are not constants but functions of coordinates.

2. ABOUT MAR

Hence, there arises a challenge to extend or modify the previously existed analytical and numerical solutions of the PEC-screen wave scattering problems to the three mentioned types of imperfect thin screens. In computational electromagnetics, one of the most powerful and efficient approaches is based on the integral equations (IE). Here, the *method of analytical regularization* (MAR), i.e., a semi-inversion of the full-wave singular IE [13], is, in fact, the only one that guarantees numerical convergence. General scheme of MAR works as follows. Commonly, the boundary PEC conditions generate a singular IE of the first kind: $\hat{G}X = F$. Split the operator $\hat{G}$ into two parts, $\hat{G}_1$ and $\hat{G}_2$. Provided that the former has a known inverse $\hat{G}_1^{-1}$, the original equation can be converted to the second-kind one: $X + \hat{A}X = B$, where $\hat{A} = \hat{G}_1^{-1}\hat{G}_2$ and $B = \hat{G}_1^{-1}F$. However, this scheme is mathematically justified only if the operator $\hat{A}$ is compact, i.e., $||\hat{A}|| < \infty$ in a certain functional space. This implies inherently that the inverted operator $\hat{G}_1$ must be a singular one while $\hat{G}_2$ is regular. It is possible to point out several different ways of extracting out an invertible singular part of original equation. It corresponds to either *canonical-shape* or to the *high-frequency* or to the *static* part of the full-wave IE operator [13]. Once it has been done, it is guaranteed, thanks to the Fredholm theorems, that the usual discretization schemes converge to the exact solution in the point-wise sense. Here, the convergence is understood as a possibility to minimize the computation error to machine precision by solving progressively larger matrices.

The variety of problems solved by MAR with the static part inversion covers a wide class of PEC zero-thickness screens [13]. Among them there are single strips and strip gratings, strip irises in a waveguide, periodic circular waveguides, open circular cylindrical screens and collections of them, longitudinally slitted infinite cone, circular disk, spherical cap, finite circular hollow pipe, etc. Any of the listed problems is reduced first to a single singular IE or a coupled pair of IEs of the first kind. A limit form of IE, corresponding to the static problem, can be inverted analytically based on the theory of the Cauchy integrals. Application of this result to full-wave IE leads to an IE of the Fredholm 2-nd kind with a smooth kernel. Hence, the existence of the unique solution is guaranteed. Numerical solution is then easy to obtain by using any reasonable discretization scheme, and the validity of the matrix truncation is justified. As the accuracy of computations is improved by increasing the number of equations and is limited only by the digital precision of computer used, MAR can be called a "numerically exact" approach. The number of equations needed for a practical accuracy of 3-4 digits is normally slightly greater than electrical size of the PEC scatterer. As we shall see, in the 2-D case of the H-polarization, these solutions can be directly extended to resistive scatterers as well because non-zero R does not change the static limit of IE. In the E-polarization case, the situation is different. Here, non-zero R changes the static behavior of the scatterer. However, in the E-case, second-kind IE obtained from the imperfect boundary condition is already a Fredholm one, whose operator vanishes in the static limit. For the axisymmetric 3-D screens, the same is valid with respect to IEs for two potential functions, which are frequently taken as E_ϕ and H_ϕ. In each case, the obtained second-kind equations correspond to the static-part inversion. It is also necessary to note that if there exists a pair of one-to-one mappings $X = CZ$, $Z = C^{-1}X$, then one can build a MAR analysis on the discretization of equivalent operator equation $Z + (C^{-1}AC)Z = C^{-1}B$, which is also a Fredholm second-kind one in the corresponding space. For example, operators C and C^{-1} can be direct and inverse integral Fourier transforms (in the single strip scattering) or discrete Fourier transforms (in the strip grating scattering) or integral Hankel transforms (in the disk scattering).

3. SCATTERING BY IMPERFECT SCREENS

Resistive strip. Consider an example in 2-D: the scattering of a given time-harmonic ($\sim e^{-i\omega t}$) field by a resistive strip [9,14,15], whose contour of cross-section is an open curve M in the plane (x, y). The scattered field has to satisfy the Helmholtz equation off M, boundary conditions on M, edge condition near the sharp edges of the screen, and the radiation condition at infinity. In the case of the H-polarization, the role of potential function is played by the component $H = H_z$. Then, the boundary conditions (1), (2) take the form as

$$\frac{1}{2}\left(\frac{\partial H^+}{\partial n} + \frac{\partial H^-}{\partial n}\right) + ik_0 Z_0 R(H^+ - H^-) = 0, \quad \partial H^+/\partial n - \partial H^-/\partial n = 0, \tag{5}$$

After decomposing the total field into the sum of the incident H^{in} and scattered one H^{sc} and presenting the latter in the form of a double-layer potential, one obtains a hyper-singular IE of the second kind:

$$ik_0 R X(\vec{r}) + \frac{\partial}{\partial n}\int_M X(\vec{r'})\frac{\partial}{\partial n'}G_0(\vec{r},\vec{r'})d\vec{r'} = -\frac{\partial H^{in}(\vec{r})}{\partial n}, \tag{6}$$

where $G_0 = i/4H_0^{(1)}(k|\vec{r}-\vec{r'}|)$ is the 2-D free-space Green's function, and $X = H^+ - H^-$ is the unknown surface-current density. Note that in (5) and hence in (6), the term containing the product $k_0 R$ is a simple perturbation to the PEC boundary condition and the IE, respectively. That is why analytical regularization of (6) can be done in the same way as for a PEC screen, and a smooth passing to the limits $k_0 \to 0$ and $R \to 0$ is possible at every step of this procedure. The inversion of the static part of (6) is based on the *diagonalization* of the integral operator with a hyper-type singularity. This is due to the existence of a set of orthogonal eigenfunctions of the IE static limit: for example, if M is a straight interval, i.e., if the strip is flat, they are the weighted Chebyshev polynomials. Further details of this analysis (in transform domain) can be found in [14,15], for a circularly curved resistive strip and for a flat resistive strip, respectively.

In the alternative case of the E-polarization, the role of potential function is played by the component $E = E_z$. Then, resistive boundary conditions (1), (2) take the form as

$$\frac{1}{2}\left(\frac{\partial E^+}{\partial n} - \frac{\partial E^-}{\partial n}\right) + \frac{ik_0}{Z_0 R}(E^+ + E^-) = 0, \quad E^+ - E^- = 0, \tag{7}$$

These conditions, together with the single-layer representation of the scattered field E^{sc}, lead us to the following IE of the second kind:

$$Y(\vec{r}) + \frac{ik_0}{R}\int_M Y(\vec{r'})G_0(\vec{r},\vec{r'})d\vec{r'} = -\frac{ik_0}{R}E^{in}(\vec{r}), \tag{8}$$

where $Y = \partial E^+/\partial n - \partial E^-/\partial n$ is the unknown surface-current density. IE (8) has a logarithmic-singular kernel G_0, hence this IE is of the Fredholm second-kind, provided that $R \neq 0$. That is why it can be discretized by using any usual discretization scheme with local or global-basis expansion functions. One can see that the norm of this IE operator is finite for any $R \neq 0$ and proportional to k_0. Therefore, it may be stated that (8) is based on the analytical inversion of the static limit of the full-wave scattering problem. However, unlike the H-case, for any $R \neq 0$ the static limit of the solution to (8) is identical zero; besides, for a fixed k_0, there is no smooth passing to the limit $R \to 0$. Still besides, one can see that the ratio $ik_0^{-1}R$ plays the role of a *Lavrentyev regularization parameter* [16].

In the case of a circularly curved open resistive strip (Fig. 1), IEs (6) and (8) can be transformed with discrete Fourier transform, and further static-part inversion can be done in the transform domain [14]. The radar cross-sections of such a strip illuminated by a plane wave [14] are presented in Figs. 2 and 3. In the flat-strip case, IEs can be tranfrormed into the Fourier transform domain. Static part inversion can be done in the transform domain as well (see [15]), with the Bessel functions as a set of transformed orthogonal eigenfunctions. The algorithms based on the space-domain and transform-domain MAR are equally high-efficient. A final remark can be done about the extension of MAR solution to a strip with the resistivity varying along the contour M. In this case, R in IE (6) and (8) must be viewed as a function of $\vec{r}$. This circumstance does not change the basic properties of IEs, and hence the same MAR schemes work out, although the rate of convergence gets worse. Based on such a modified algorithm, a cylindrical reflector antenna with a variable-resistivity edge loading was analysed in [17].

Material and impedance strips. In the homogeneous material-strip scattering, we start from conditions (1), (2) with $W = 0$. Together with the representation of the scattered field as a sum of a single- and a double-layer potentials, they yield now not one but two second-kind *decoupled* IEs. Each of the latter can be further treated in the same way as it was done previously for a resistive strip scattering with an H-wave and an E-wave incident, respectively. In the E-polarization case, material thin-strip boundary conditions lead us to a similar pair of the second-kind decoupled IEs, with the parameters R and S interchanged. Hence, in the *material-strip* scattering, a difference in the electromagnetic behavior between the E-wave and H-wave cases vanishes. Note that, to build the scattered field, the contributions from the solutions of the both IEs must be taken into account. Material strip scattering was analyzed in [18] where the 1-st order conditions and Fourier-transform version of IEs were used. In the scattering by a multilayer material strip or an impedance strip, the boundary conditions (1), (2) or, equivalently, (3) should be used. They bring us to the pair of *coupled* IEs of the second kind, where cross-resistivity W plays the role of the coupling parameter. If the surface impedances are the same: $Z^+ = Z^-$, then $R = \frac{1}{2}Z^+$, $S = (2Z^+)^{-1}$, $W = 0$, and hence the IEs again decouple. Special case of $Z^+ = -Z^-$ can be considered as well [4].

Imperfect strip gratings. In the case of the scattering of plane waves by a flat grating made of identic periodically spaced resistive, material or impedance strips of period p (Fig. 4), the same MAR approach as for a single strip can be used. This is due to the fact that in the kernels of corresponding IEs, the quasi-periodic free-space Green's function, G_p, takes the place of G_0. As $G_p = G_0 + P$, where P is a regular function at $\vec{r} \rightarrow \vec{r'}$, the singularities are kept the same, and hence a similar static-part inversion results in the regularized infinite-matrix equations. A grating of flat resistive strips has been considered in [10,19-22]. Note that in the H-wave case, the results published in [19] were erroneous as the matrix elements did not decrease with greater indices. This is because no regularization of a hyper-singular IE was performed. The latter was developed in [20,21], where it was noted that the same mistake was characteristic for the other papers considering the scattering of H-waves by a resistive strip grating. In the E-wave case, the results published in [10,19] are correct and agree with the data of [20-22] where different discretization was used. In Figs. 5 and 6, presented are the power fractions for the plane wave scattering by a resistive strip grating [20]. In Figs. 7 and 8, the same is presented for the scattering by a thin-strip dielectric grating considered in [20]. Note that the latter results were confronted with the exact solution of the "domain" IE for a finite-thickness dielectric strip grating. Comparison showed a very good agreement for the strip thickness being 0.1 of the period.

Imperfect disk. Consider a curved rotationally symmetric disk supporting resistive boundary conditions. Here, two simplest diffraction problems arise: excitation of the disk by either a coaxial vertical electric dipole (CVED) or a magnetic one (CVMD). The field in such a geometry is ϕ-independent and can be expressed via a single potential function: H_ϕ or E_ϕ, respectively [23]. In the case of electric (or magnetic) dipole, we arrive at the singular IE of the second kind similar to (6) (or (8)), with the ϕ-independent scalar 3-D

Green's function K_0 taking place of G_0:

$$K_0(\vec{r}, \vec{r'}) = r' \int_0^{2\pi} \frac{e^{ikR}}{R} \cos\psi d\psi, \tag{9}$$

where

$$\psi = \phi - \phi', \qquad R = [r^2 + r'^2 + 2rr' \cos\psi + (z - z')^2]^{1/2} \tag{10}$$

The domain of integration in IE correspondingly changes to an open curve in the halfplane $(r \geq 0, z)$. Note that the function K_0 has the same logarithmic singularity as G_0, if $\vec{r} \to \vec{r'}$ [23, p. 67]. In the CVED-case, regularization is needed to reduce the problem of the disk scattering to the infinite-matrix equation of the Fredholm second-kind. This is done by applying a Galerkin scheme with the special Jacobi polynomials, which form the set of orthogonal eigenfunctions of IE static limit. In the CVMD-case, the basic IE is immediately of the Fredholm second kind provided that $R \neq 0$, and may be discretized via any reasonable projection scheme. In Fig. 9, presented are the frequency scans of the power fractions related to the CVED-excited resistive flat-disk antenna ontop of a grounded dielectric substrate [24].

In the case of a thin flat material disk of high dielectric and magnetic constants, the boundary conditions (1) and (2) yield a coupled set of two IEs. Each of them is analogous to one of the *resistive-disk* IEs, with additional regular interaction term. Therefore, regularization and discretization is done as above. In [24,25], these IEs have been transformed to the Hankel-transform domain and converted to the dual IEs for the surface-current transforms. Expansion functions are then transformed to the special-type Bessel functions and the Laguerre functions, depending on the type of IE, and lead to efficient numerical solutions.

4. CONCLUSIONS

It is possible to modify the MAR solutions, previously developed in the PEC-screen scattering, to the imperfect thin screens: resistive, material, and impedance ones. This opens a way for a numerically exact analysis of not only the scattering but also the absorption of waves by screens. Some preliminary results can be found in [14,17,20,24,25].

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