Bicomplex electromagnetic waves in scattering and diffraction problems

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Abstract *The mathematical theory of bicomplex electromagnetic waves in two-dimensional scattering and diffraction problems is developed. The Vekua's integral expression for the two-dimensional fields valid only in the closed source-free region is generalized into the radiating field. The boundary-value problems for scattering and diffraction are formulated in the bicomplex space. The complex function of a single variable, which obeys the Cauchy-Riemann relations and thus expresses low-frequency aspects of the near field at a wedge of the scatterer, is connected with the radiating field by an integral operator having a suitable kernel. The behaviors of this complex function in the whole space are discussed together with those of the far-zone field or the amplitude of angular spectrum. The Hilbert's factorization scheme is used to find out a linear transformation from the far-zone field to the bicomplex-valued function of a single variable. This transformation is shown to be unique.*

1. Introduction

This paper is a concise version of the full length manuscript submitted for publication in the journal, which is written as a supplement to the early works [1][2][3] on the new approach in electromagnetic field problems. The fundamental concept through those works is based on the bicomplex or bivector, describing four-dimensional quantities in terms of position vectors in a two-dimensional space. Thus the problems are strictly limited to two-dimensional cases. This bicomplex treatment is applied to solve the boundary-value problems.

The terminology "bivector" was first used in the Silverstein's papers [4] and [5] to describe a linear combination $\boldsymbol{E} + i\boldsymbol{H}$ of a time-dependent electric field vector $\boldsymbol{E}$ and a corresponding magnetic field vector $\boldsymbol{H}$. For time-harmonic fields varying with $\exp(i\omega t)$, Bateman [6] introduced another unit $\pm\, i$ to combine $\boldsymbol{E}$ and $\boldsymbol{H}$ as $\boldsymbol{E} \pm i\boldsymbol{H}$. Bremmer [7] and Roubine and Bolomey [8] replaced the Bateman's ambiguity symbol $\pm\, i$ by j and generalized a single-complex number to a double-complex number named "bicomplex number". These combined electromagnetic fields in chiral or bianisotropic media are extensively discussed by Lakhtakia [9][10].

The bicomplex fields studied in [1][2][3] are also the time-harmonic fields varying with $\exp(j\omega t)$ as usual, but are not the combined fields mentioned above. They are described in terms of the two complex position-parameters, $x + iy$ and $x - iy$. The resulting fields are mathematically treated as the complex elements of commutative algebra in the four- dimensional space $(1,i,j,ij)$ [8] and are thus expressed as the functions of four independent units $1,i,j,ij$. Calculations of the integrals and the series are performed over the bicomplex plane.

The field is represented in an integral form similar to the Vekua's integral [11] that holds valid only for the bounded space. Extensions were made in [1] to apply the formula to the open space as in scattering or diffraction problems. Our bicomplex fields are, therefore, outgoing waves that satisfy the Sommerfeld's radiation condition. Some examples were illustrated to demonstrate the principal idea of this extension [1][2]. The far-zone field or the radiation pattern (the angular-spectrum distribution) was factorized into a factor and its complex conjugate. These factors were studied in detail in connection with the Vekua's Φ-function [1]. Subsequently, the Hilbert's technique [12] was adopted to accomplish the factorization [3]. The present paper is intended partly to acquaint us with the generic theory of bicomplex electromagnetic waves in the boundary-value problems, as well as to supplement the theory, presenting new results.

2. Bicomplex Algebra and Integral Representation for Solutins

In this section, we study some algebraic rules for quantities that contain two imaginary units i and j. We define these numbers by the equations

$$j^2 = -1\,, \quad i^2 = -1 \qquad (1)$$

and apply the usual commutative rules of algebra for every number; whenever j^2 and i^2 appear, we replace them by -1. We also apply the word "imaginary" as used in the complex analysis for i and by no means apply it for j. The quantity j is simply treated like a real number, except for the rule in Eqs.(1).

The complex variables used in the rectangular coordinate system (x,y) are denoted by z and ζ where

$$z = x + iy\,, \quad \zeta = x - iy \qquad (2)$$

or, by polar coordinates r, θ on the complex plane,

$$z = re^{i\theta} \qquad (3)$$

and ζ is the complex conjugate of z, which is simply denoted by the conventional symbol

$$\zeta = \bar{z} \qquad (4)$$

We note that the Hankel function of the second kind of w $(= u + iv)$, which will appear in this paper, is a bicomplex function

$$H_0^{(2)}(w) = J_0(w) - jY_0(w)$$

Let us now consider a sum of plane waves in the polar coordinate system

$$u(z,\zeta) = i\int_{-i\infty + ij\pi/2 + \alpha_2}^{+i\infty - ij\pi/2 + \alpha_1} \Pi(\theta')\, e^{-jkr\cos(\theta - \theta')} d\theta' \qquad (5)$$

where αs are constants whose values are real and satisfy

$$\left|\theta - \alpha_{1,2}\right| < \pi/2 \qquad (6)$$

The expression (5) is a generalization of the representation of plane wave spectrum given in [13]. This is readily verified by replacing j by i in Eq.(5). Changing the variable of integration from θ' to ξ' with $\xi' = i(\theta' - \theta)$ and $\theta - \alpha_{1,2} \equiv \theta_{1,2}$, we have

$$u(z,\zeta) = \int_{+\infty - j\pi/2 - i\theta_2}^{-\infty + j\pi/2 - i\theta_1} \Pi(\theta - i\xi')\, e^{-jkr\cosh\xi'} d\xi' \qquad (7)$$

where $\Pi(\theta)$ is a spectrum function which is closely related to the far-zone field of u. In far zone, the main contribution of the angular spectrum of plane waves can be isolated as follows.

$$u(z,\zeta) \simeq \pi j\sqrt{\frac{2}{\pi kr}}\, e^{-jkr + j\frac{\pi}{4}}\, \Pi(\theta) \qquad (8)$$

This is verified mathematically when the method of stationary phase [13] is applied to the bicomplex space. From both physical and mathematical points of view, the one-to-one correspondence between u and Π is evident. Thus the angular spectrum can uniquely be determined from the boundary data imposed as the initial conditions if the given conditions are well-posed (see [14]).

3. Factorization of the Spectrum Function *Π(θ)*

Apart from Eq.(7), we begin by factorizing *Π(θ)* into a product of the form

$$\Pi(\theta) = \pi^{+}(\theta)\ \pi^{-}(\theta) \tag{9}$$

where

$$\pi^{-}(\theta) = \overline{\pi^{+}(\theta)} \tag{10}$$

and the bar denotes complex conjugate with respect to the imaginary unit *i* as in Eq.(4). On the other hand, this operation does not affect the unit *j*, so that the sign of *j* remains unchanged. The problem stated above is reduced to the simplest example of the Riemann-Hilbert problems [12; §40] with a few modifications.

Following the Hilbert's theory of factorization [12][15], the symbol + or – is referred to, respectively, as an internal or external region into which $\pi^{+}(\theta)$ or $\pi^{-}(\theta)$ is continued analytically. The boundary between these internal and external regions is chosen as a unit circle (r =1) on the z-plane. Thus $\pi^{+}(\theta)$ and $\pi^{-}(\theta)$ may be written

$$\pi^{-}(\theta) = \psi(w) \equiv \frac{1}{2\pi i}\oint_{\gamma} e^{wt}\ \Psi(t)dt \tag{11}$$

and

$$\pi^{+}(\theta) = \overline{\psi\left(\frac{(jk/2)^2}{\overline{w}}\right)} \tag{12}$$

where

$$w = \frac{jk}{2\,e^{i\theta}} = j\frac{k}{2}e^{-i\theta} \tag{13}$$

The function *Ψ(z)* in Eq.(11) must be analytic outside the obstacle we consider as a scatterer and must vanish at infinity with order of $1/z$. All singularities of *Ψ(z)* are located inside the obstacle. The γ is a contour that surrounds the obstacle on the z-plane and the integration is carried out along γ in a positive direction (in a counterclockwise direction) (see Fig.1). The transform from *ψ(w)* to *Ψ(z)* may also be possible by the Laplace integral

$$\Psi(z) = \int_{0}^{\infty \exp(-i\theta_0)} e^{-wz}\ \psi(w)\,dw \tag{14}$$

where θ_0 is a real constant suitably chosen so that the integral converges. For example, if z is given by Eq.(3), then θ_0 must be θ.

In general, the spectrum function *Π(θ)* has zeros over the interval between 0 and 2π. They correspond to null points in the far-zone field pattern. The Hilbert's technique will fail at these points because the logarithmic function ln *Π(θ)* will diverge. To regularize the problem, a small quantity $+ j\varepsilon$ or $- j\varepsilon$ that is treated as a "real number" is added to *Π(θ)* [3]. By doing this, the function ln *Π(θ)* becomes regular over the interval $[0, 2\pi]$ and can be decomposed into a sum of ln $\pi^{+}(\theta)$ and ln $\pi^{-}(\theta)$, each of which is analytically continued to the internal ($|z|<1$) and external ($|z|>1$)

regions, respectively. We note that these decomposed factors are uniquely determined [3] except for the ambiguity arising from the two possibilities of choice for the sign of $j\varepsilon$; namely, the replacement of j by $-j$ leads to another solution, but this ambiguity absolutely disappears in the final expressions for u. Therefore, both choices provide the same solution for physical quantities.

To show this briefly, we quote an example $\Pi(\theta) = \cos\,\theta$ given in [3]. By adding a small quantity $\pm j \sinh\,\varepsilon$ instead of $\pm j\varepsilon$, we have

$$\begin{aligned}\Pi(\theta) &= \cos\,\theta \pm j\,\sinh\,\varepsilon \\ &= \left(1 \mp j e^{-\varepsilon} e^{-i\theta}\right)\left(\tfrac{1}{2} e^{i\theta} \pm \tfrac{j}{2} e^{\varepsilon}\right)\end{aligned} \tag{15}$$

Thus

$$\pi^{-}(\theta) = e^{i\Theta} \sqrt{\pm \tfrac{j}{2} e^{\varepsilon}}\,\left(1 \mp j e^{-\varepsilon} e^{-i\theta}\right) \tag{16}$$

where Θ is an arbitrary real constant. For the more general approach, the reader should refer to [3]. With $\varepsilon \to 0$,

$$\psi(w) = e^{i\Theta} \sqrt{\pm \tfrac{j}{2}}\,\left(1 \mp \tfrac{2}{k}\, w\right) \tag{17}$$

The calculation of the integral in Eq.(14) is straightforward. The result is

$$\Psi(z) = e^{i\Theta} \sqrt{\pm \tfrac{j}{2}}\,\left(\tfrac{1}{z} \mp \tfrac{2}{kz^2}\right) \tag{18}$$

It will be shown later that the ambiguity of sign $\pm$ does not appear in the field expressions (see Eqs.(39) and (48)).

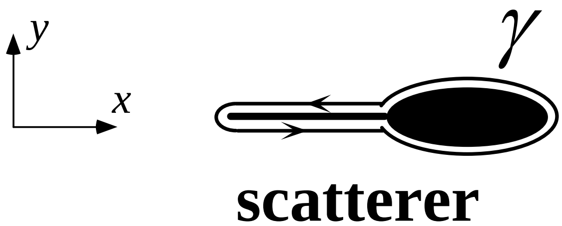

Fig.1 Geometry of an example scatterer on the xy plane. The γ is a path of integration on the z-plane.

4. Bicomplex Wave Solutions

We substitute the results obtained in Section 3 into the angular spectrum (7). Then we have

$$\begin{aligned} u(z,\zeta) = &\int_{+\infty - j\pi/2 - i\theta_2}^{-\infty + j\pi/2 - i\theta_1} e^{-jkr\cosh\xi'} \\ &\times \frac{1}{2\pi i} \oint_\gamma e^{j\frac{kt}{2} e^{-i\theta - \xi'}}\, \Psi(t)\, dt \\ &\times \overline{\frac{1}{2\pi i} \oint_\gamma e^{j\frac{kt}{2} e^{-i\theta + \bar{\xi}'}}\, \Psi(t)\, dt}\;\, d\xi' \end{aligned} \tag{19}$$

The following new function and new variable are used to simplify the expression.

$$\Phi(\zeta) = \overline{\Psi(\bar{\zeta})} \qquad \tau = \bar{t} \tag{20}$$

Thus

$$u(z,\zeta) = \int_{+\infty - j\pi/2 - i\theta_2}^{-\infty + j\pi/2 - i\theta_1} e^{-jkr\cosh\xi'}$$
$$\times \frac{1}{2\pi i}\oint_{\gamma} e^{j\frac{kt}{2}e^{-i\theta-\xi}}\ \Psi(t)\,dt$$
$$\times \frac{1}{2\pi i}\oint_{\bar{\gamma}} e^{j\frac{k\tau}{2}e^{i\theta+\xi}}\ \Phi(\tau)\,d\tau\ \ d\xi' \qquad (21)$$

Changing the order of integration and further using the mathematical identity for the Hankel function [2] ,

$$H_0^{(2)}(k\sqrt{(z-t)(\zeta-\tau)})$$
$$= \frac{j}{\pi}\int_{-\infty + j\pi/2 - i\theta_1}^{+\infty - j\pi/2 - i\theta_2} e^{-jkr\cosh\xi}$$
$$\times e^{j\frac{kt}{2}e^{-i\theta-\xi} + j\frac{k\tau}{2}e^{i\theta+\xi}} d\xi' \qquad (22)$$

with

$$\theta_1 = -\arg\sqrt{1-t/z},\ \theta_2 = \arg\sqrt{1-\tau/\zeta}$$

we finally arrive at

$$u(z,\zeta) = \frac{1}{(2\pi i)^2}\oint_{\gamma} dt \oint_{\bar{\gamma}} d\tau\ K(z,\zeta;t,\tau)\ \Psi(t)\Phi(\tau) \qquad (23)$$

where

$$K(z,\zeta;t,\tau) = \pi j H_0^{(2)}(k\sqrt{(z-t)(\zeta-\tau)}) \qquad (24)$$

The original problem now becomes the problem of finding out the analytic function $\Psi(z)$. The function $\Phi(\zeta)$ associated with $\Psi(z)$ by Eqs.(20) is also analytic outside the conjugate contour $\bar{\gamma}$ (denoted by the bar) on the ζ-plane. However, the integration of $\Phi(\zeta)$ is carried out along $\bar{\gamma}$ in a "positive direction" or in a counterclockwise direction. We emphasize again that the double integration in Eq.(23) must be carried out in a positive sense.

5. Current/Charge Distributions over the Surface

The expression for the transverse electric and magnetic fields is written in a differential form. The use of the differential operator $\partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y)$ yields

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{1}{2} j\omega\mu(H_y + iH_x) \qquad \text{for E waves}\\ &= -\frac{1}{2} j\omega\varepsilon(E_y + iE_x) \qquad \text{for H waves}\end{aligned} \qquad (25)$$

where ω is the angular frequency, and μ and ε are the permeability and permittivity of the medium. The integral representation is simply to differentiate K with respect to z in Eq.(23). The derivative contains the Cauchy kernel $1/(z-t)$ and the remaining part has logarithmic singularities which are much weaker than the Cauchy kernel. For this principal part, we apply the Cauchy integral formula. Then we have

$$\frac{\partial u}{\partial z} = \overline{a_0}\ \Psi(z) - \left(\frac{k}{2}\right)^2 \frac{1}{(2\pi i)^2} \oint_{\gamma} dt \oint_{\bar{\gamma}} d\tau$$
$$\times (\zeta - \tau) Q(k\sqrt{(z-t)(\zeta-\tau)})\, \Psi(t) \Phi(\tau) \qquad (26)$$

where

$$Q(x) = \frac{1 + j\pi\left(\frac{x}{2}\right) H_1^{(2)}(x)}{\left(\frac{x}{2}\right)^2} \qquad (27)$$

and

$$a_0 = \frac{1}{2\pi i} \oint_{\gamma} \Psi(t) dt$$
$$\overline{} = \frac{1}{2\pi i} \oint_{\bar{\gamma}} \Phi(\tau) d\tau \qquad (28)$$

If the double integral is bounded in the limit of $k \to 0$, the second term on the right hand side of Eq.(26) tends to zero because the factor $k/2$ squared in front of the integral vanishes. Thus the first term describes static behaviors of the transverse field. In general, however, the integral is not bounded with this static approximation. In such a case, the integral (the second term in Eq.(26)) contributes to the low-frequency field, so that ignoring the integral leads to an erroneous solution. The full expression for $\partial u/\partial z$ is therefore exact and valid over the surface of the scatterer.

The electric current density J_s or σ_s charge density induced over the surface is given by the tangential component of the magnetic field H or the normal component of the electric field E, respectively, all of which are described in terms of the unit normal n and the complex conjugate $\bar{n}$ as follows.

$$n = n_x + in_y\ , \quad \bar{n} = n_x - in_y \qquad (29)$$

$$n\frac{\partial u}{\partial z} = \frac{1}{2} j\omega\mu(H_t + iH_n) \qquad \text{for E waves}$$
$$= -\frac{1}{2} j\omega\varepsilon(E_t + iE_n) \qquad \text{for H waves} \qquad (30)$$

Taking the real part of Eq.(30) for E waves and the imaginary part for H waves, we obtain the surface current and the surface charge, respectively.

6. Boundary-Value Problems

In this section, we do not discuss about how to determine $\Pi(\theta)$ from the boundary data. Our aim is to give a proof for the uniqueness of a solution. This is based on the consequence of one-to-one correspondence between the field u and the spectrum function Π. The integral equation for a generalized boundary condition specified on the boundary surface will be established by using

$$jkr \cos(\theta - \theta') = j\frac{kz}{2} e^{-i\theta'} + j\frac{k\zeta}{2} e^{i\theta'}$$

in Eq.(5) and applying

$$\frac{\partial}{\partial n} = n\frac{\partial}{\partial z} + \bar{n}\frac{\partial}{\partial \zeta}$$

to the boundary. The result is

$$C = Au + B\frac{\partial u}{\partial n}$$
$$= \int_{+\infty - j\pi/2 - i\theta_2}^{-\infty + j\pi/2 - i\theta_1} e^{-jkr\cosh\xi'}$$
$$\times \left[A - j\frac{kB}{2}\left(n\, e^{-i\theta - \xi'} + \bar{n}\, e^{i\theta + \xi'}\right)\right]$$
$$\times \Pi(\theta - i\xi')\, d\xi' \tag{31}$$

where z or ζ is a point on the boundary or the conjugate boundary, respectively. From the uniqueness theorem [18], only one solution for u that satisfies Eq.(31) is obtained. As already mentioned, u corresponds to Π one-to-one. Thus we have proved that the spectrum function Π is uniquely determined.

Since the function Π describes a far-zone field pattern by Eq.(9) or by Eqs.(11) and (12), the problem is now reduced to the Hilbert problem presented in Section 3 in which Π is given on the unit circle $|s| = 1$ to determine the analytic fuction Ψ. The two possible solutions of the nonlinear integral equation

$$\Pi(\theta) = \left| \frac{1}{2\pi i} \oint_\gamma e^{j(\frac{k}{2}e^{-i\theta})t}\, \Psi(t)dt \right|^2 \tag{32}$$

have the sign ambiguity $\pm j$ but these all provide the unique field u. In this respect, the obtained function Ψ is said to be a uniquely determined solution.

7. Diffraction by an Infinite Half-Plane

In order to illustrate the use of the bicomplex theory developed, and to gain some insight into the framework of the analysis, we consider a canonical problem of diffraction by a half-plane. In this example, the perfectly conducting plane lies on the x axis from zero to infinity. The paths of integration on the z- and ζ-planes start from a point at infinity, go to the origin, round it in a positive direction, and again return to infinity. The incident wave is assumed to come from infinity, making an angle φ with the x axis.

We here discuss the physical optics (PO) solution for E waves. The electric current along the half plane is approximated by a truncation of the current induced over the infinite plane [20].

$$J_s(x) = \frac{2k}{\omega\mu} \sin\varphi\, e^{jkx\cos\varphi} \tag{33}$$

The corresponding spectrum function is given by

$$\Pi(\theta) = -\frac{1}{2\pi i}\int_0^\infty \left[-ij\,\frac{\omega\mu}{2} J_s(x)\right] e^{jkx\cos\theta}\, dx$$
$$= -\frac{1}{2\pi}\,\frac{\sin\varphi}{\cos\theta + \cos\varphi} \tag{34}$$

The factorization for $\Pi(\theta)$ is not difficult even if it is performed by trial (refer to [1]). Our result is

$$\pi^-(\theta) = \psi(w) \equiv \frac{1}{2\pi i}\oint_\gamma e^{wt}\, \Psi(t)dt$$

$$= e^{i\Theta}\sqrt{\frac{-\sin\varphi}{\pi}}\,\frac{e^{j\frac{\varphi}{2}}}{e^{-i\theta}+e^{j\varphi}}$$

$$= e^{i\Theta} jk\,\upsilon\sqrt{\frac{-\sin\varphi}{\pi}}\,\frac{1}{w+j2k\upsilon^2} \qquad (35)$$

$$(0<\varphi<\pi)$$

where

$$\upsilon = \frac{1}{2}e^{j\frac{\varphi}{2}} \qquad (36)$$

The calculation of the integral of Eq.(14) is straightforward.

$$\Psi(z) = e^{i\Theta}\,jk\,\upsilon\sqrt{\frac{-\sin\varphi}{\pi}}\;e^{j2kz\upsilon^2}\,E_1(j2kz\upsilon^2) \qquad (37)$$

where $E_1(z)$ is the exponential integral defined below.

$$E_1(z) = \int_z^{\infty}\frac{e^{-t}}{t}\,dt$$

$$= -0.57721\ldots - \ln z - \sum_{n=1}^{\infty}\frac{(-z)^n}{n\,n!} \qquad (38)$$

The above many-valued function has a branch cut running from zero to infinity along the x axis and jumps by $i2\pi$ when the point z moves across the x axis along the y axis. We see that only this jump contributes to the calculation of $\nu(z)$:

$$\nu(z) = \Psi(z+i0) - \Psi(z-i0)$$

Indeed, in Eq.(37), we simply replace E_1 by $i2\pi$. Thus by taking into account $a_0 = \psi(0)$, we obtain

$$\overline{a_0}\;\nu(x) = -ij\,k\sin\varphi\,e^{j2kx\upsilon^2} \qquad (39)$$

which, in the limit of $k\to 0$, is now expected from Eq.(26) to represent a good approximation to the exact solution

$$\left(\frac{\partial u}{\partial z}\right)^{+} - \left(\frac{\partial u}{\partial z}\right)^{-} = -ij\frac{\omega\mu}{2}J_s(x)$$

$$= -ij\,k\sin\varphi\,e^{jkx\cos\varphi}$$

$$= -ij\,k\sin\varphi\,e^{j2kx\cos^2(\varphi/2)}e^{-jkx} \qquad (40)$$

We then find that the u is in place of $\cos(\varphi/2)$ except for a factor $\exp(-jkx)$. In fact, we are able to demonstrate the formal equivalence between the approximate solution and the exact one by

$$\overline{a_0}\;\nu(x) \leftrightarrow -ij\,\frac{\omega\mu}{2}J_s(x)\,e^{jkx}$$

$$w \leftrightarrow -j2k\sin^2(\theta/2) \qquad (41)$$

$$\upsilon \leftrightarrow \cos(\varphi/2)$$

To further understand the use of the present theory, it is helpful to consider another specific example. We consider a *fringe wave diffraction* by an infinite half-plane. The fringe wave or edge wave is defined as a correction of the physical optics approximation [22]. The spectrum function can be computed by subtracting the approximate solution from the exact solution presented in the James' book [20] (his notation D for the edge diffraction coefficient must be multiplied by $-1/4\pi$). Thus for the fringe waves in case of E waves

$$\Pi(\theta) = \frac{1}{2\pi} \frac{\sin\frac{\varphi}{2}}{\sin\frac{\theta}{2} + \cos\frac{\varphi}{2}} \tag{42}$$

First let us factorize *Π(θ)* into $\psi(w)$ and its conjugate. This factorization is very similar to that for Eq.(340). Thus

$$\psi(w) = \frac{e^{i\Theta}}{\sqrt{2\pi}} \frac{\sqrt{k/2}\,(2\upsilon + ij)}{\sqrt{w} - i\sqrt{j2k}\,\upsilon} \tag{43}$$

where Θ is an arbitrary real constant. By substituting the above expressions into Eq.(14) and changing the variable of integration, we have

$$\Psi(z) = \frac{e^{i\Theta}}{\sqrt{2\pi}} \sqrt{\frac{k}{2}}\,(2\upsilon + ij)\, f(z) \tag{44}$$

where

$$f(z) = \sqrt{\frac{\pi}{z}} + i2\sqrt{j2k}\,\upsilon \int_0^\infty \frac{e^{-u^2}du}{u - i\sqrt{j2kz}\,\upsilon} \tag{45}$$

The exact solution in this case is known [23]:

$$\begin{aligned}\left(\frac{\partial u}{\partial z}\right)^+ - \left(\frac{\partial u}{\partial z}\right)^- &= -ij\frac{\omega\mu}{2} J_s(x) \\ &= \frac{2}{\pi i}\sqrt{\frac{jk}{2}} \sin\frac{\varphi}{2} \Big[\sqrt{\frac{\pi}{x}}\, e^{-jkx} \\ &\quad - j2\sqrt{2\pi k}\cos\frac{\varphi}{2}\, e^{jkx\cos\varphi} F\left(\sqrt{2kx}\cos\frac{\varphi}{2}\right)\Big]\end{aligned} \tag{46}$$

where $F(x)$ is the complex Fresnel integral [24]

$$\begin{aligned}F(x) &= \int_x^\infty e^{-jt^2}dt \\ &= \frac{x}{\sqrt{\pi}}\, e^{-jx^2} \int_0^\infty \frac{e^{-t^2}}{t^2 + jx^2}\, dt\end{aligned} \tag{47}$$

On the other hand, from Eq.(44), we obtain

$$\begin{aligned}&\overline{a_0}\; \nu(x) \\ &= \frac{2}{\pi i}\sqrt{\frac{jk}{2}} \sin\frac{\varphi}{2} \\ &\times \Big[\sqrt{\frac{\pi}{x}} - j2\sqrt{2\pi k}\;\upsilon\, e^{j2kx\upsilon^2} F\left(\sqrt{2kx}\,\upsilon\right)\Big]\end{aligned} \tag{48}$$

which represents an approximate expression to the exact solution (46) in the low-frequency regime.

We see that the correspondence principle (41) still holds valid for the case of *E* waves. We need further investigation with the aid of more useful examples together with useful integral formulas. However, the method of comparison, described above, may be applicable to a broad class of applications if the factorization for *Π(θ)* is achieved analytically.

8. Conclusions

The general outline of the theory of bicomplex waves has been sketched for high-frequency scattering or diffraction. The high-frequency fields have been described on the basis of the analytic function of a single variable *z*, which represents low-frequency features of the transverse electric or magnetic fields. The integral operator to transform the former into the latter was also given in connection with the bicomplex description in which the solutions for the stationary fields of arbitrary nature are concisely written.

The present treatment is likely to provide a great deal of insight into the local nature of the fields, particularly of the current and charge near a wedge, prior to its final computation.

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