Surface Waves in Diffraction Gratings

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Abstract

A new general existence criterion of surface waves in diffraction grating is proposed. It is based on the consideration of the so-called augmented scattering matrix. This matrix arises if one takes into account not only oscillating Rayleigh waves but also those which grow (attenuate) far from the grating. The numerical implementation of the suggested existence criterion is discussed.

1 Introduction

It is known [1] that diffraction properties of grating can remarkably change at certain (threshold) frequencies. In a neighborhood of such frequencies an excitation of surface waves is possible. The energy of these waves is concentrated near the grating, that sometimes leads to destruction of a grating. That is why an opportunity to predict the appearance of surface waves is of great importance.

The existence criterion of surface waves in diffraction grating is proposed. It is based on the consideration of the so-called augmented scattering matrix. This matrix arises if one takes into account not only oscillating Rayleigh waves but also those which grow (attenuate) far from the grating. If such a matrix has an eigenvalue(s) equal to one then surface waves exist.

The criterion is valid for gratings of various geometry. It is briefly formulated in Section 2 (more details may be found in [3, 2]). Its numerical implementation is based on the finite elements technique in combination with some optimization ideas and is discussed in Section 3.

2 Statement of the problem and the augmented scattering matrix

It is well known that plane wave diffraction problem by a periodic structure (in two dimensions) may be reduced to a strip (or to a half-strip). The width of the strip is equal to the period of grating which may be considered to be 2π without loss of generality. Possible geometries of such strips are shown in Fig. 1.

The sought field $u(x, z)$ (for example, the E_y component of TE-polarized electromagnetic wave) satisfies the uniform Helmholtz equation in the (half)strip with the refraction index $\chi(x, z)$ which is assumed to be 2π-periodic function in x and equal

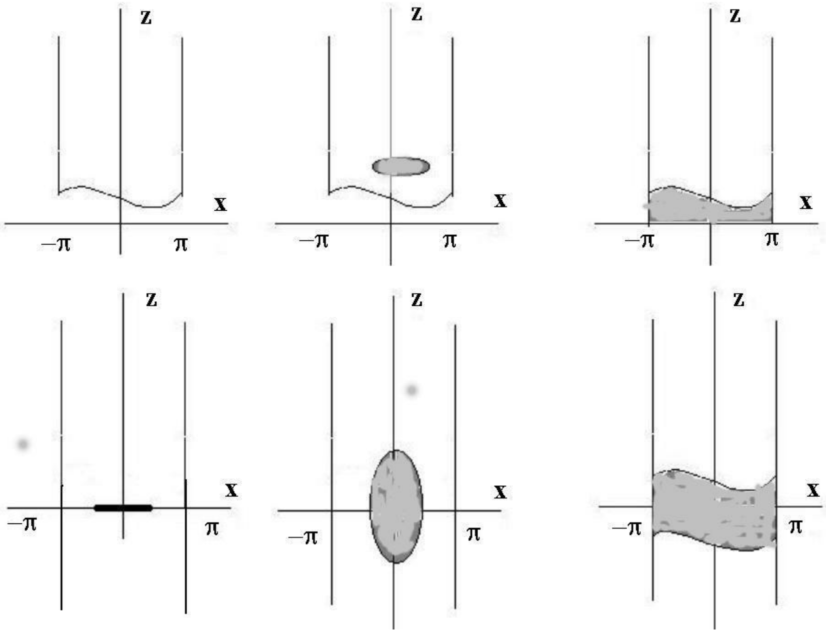

Figure 1: Possible geometry of gratings

to one for $|z| > R$ for sufficiently large R. It also satisfies the Dirichlet or Neumann boundary conditions at exterior and/or interior boundaries and usual conditions of continuity at interior interfaces. At flank boundaries of the strip the quasiperiodic conditions

$$u\big|_{x+2\pi} = e^{2\pi i\alpha}\, u\big|_{x}, \quad \partial_x u\big|_{x+2\pi} = e^{2\pi i\alpha}\, \partial_x u\big|_{x}. \tag{1}$$

are posed to take into account the incidence plane wave, $\alpha = k\sin\theta$, where k is the wave number and θ is the incidence angle.

There is a set of obvious solutions of Helmholtz equation satisfying quasiperiodic conditions if $|z| > R$ which are called Floquet (or Rayleigh) waves. To describe these waves let us introduce eigenvalues $\lambda_n = \pm(k^2 - (n+\alpha)^2)^{1/2}$ of the auxiliary problem

$$\left[\frac{d^2}{dx^2} + (k^2 - \lambda^2)\right] v(x) = 0; \quad v(x+2\pi) = e^{2\pi i\alpha} v(x), \quad v'(x+2\pi) = e^{2\pi i\alpha} v'(x)$$

and let $2T$ stand for the number of real eigenvalues, M is the number of complex eigenvalues in the strip $0 < \mathrm{Im}\lambda_{T+1} < \cdots < \mathrm{Im}\lambda_{T+M} < \gamma$ for some $\gamma > 0$. Denote by

$$U_n^{\pm}(z) = \begin{cases} \frac{e^{\mp i\lambda_n z}}{\sqrt{4\pi\lambda_n}}, & n = 1, \ldots, T, \\ \frac{1}{\sqrt{8\pi|\lambda_n|}}\left(e^{-|\lambda_n|z} \mp i e^{|\lambda_n|z}\right), & n = T+1, \ldots, T+M \end{cases} \tag{2}$$

and

$$u_n^{\pm}(x,z) = U_n^{\pm}(z) e^{i(n+\alpha)x}. \tag{3}$$

Solutions (3) corresponding to numbers $n = 1, \ldots, T$ are called incoming $(+)$ and outgoing $(-)$ Floquet waves while those with $n = T+1, \ldots, T+M$ are said to be nonuniform Floquet waves.

The existence criterion of surface waves is based on the following fundamental result ([4, 5]):

Let $V(\gamma)$ be the space of solutions of the main problem with the behaviour $u = O\left(e^{\gamma z}\right)$ as $z \to \infty$. Then $\dim V(\gamma) = T + M$ and the basis may be chosen as $\{Y_1, \ldots, Y_{T+M}\}$, where Y_m are fixed by asymptotics

$$Y_m \asymp u_m^+ + \sum_{n=1}^{T+M} S_{mn} u_n^- + u', \quad u' = o\left(e^{-\gamma z}\right) \quad \text{as} \quad |z| \to \infty. \tag{4}$$

The matrix $\|S_{mn}\|_1^{T+M}$ is called the augmented scattering matrix. It is unitary due to the normalization of solutions (2).

The existence of surface waves is equivalent to the existence of the linear combination of solutions Y_m such that $u_{sw} = \sum c_m Y_m \to 0$ as $z \to \infty$. Thus one immediately arrives to the following existence criterion. *Surface wave exists if there exists some positive γ and M depending on γ such that*

$$\det \left| S_{(22)} - I \right| = 0, \tag{5}$$

where $S_{(22)}$ is the M-by-M block of the matrix S corresponding to nonuniform Floquet waves.

3 Numerical implementation of the existence criterion

To find each (m-th) row of the scattering matrix one has to find the solution of the problem which is fixed by the asymptotics (4). To that end consider the elliptic boundary value problem for the auxiliary function u_{R_1} which should satisfy same conditions as the function u and the Neumann condition

$$\left.\frac{\partial u_{R_1}}{\partial z}\right|_{z=R_1} = \left.\frac{\partial Y_m}{\partial z}\right|_{z=R_1} \tag{6}$$

for sufficiently large $R_1 > R$. Due to linearity of this problem u_{R_1} linearly depends on coefficients S_{mn}. The solution u_{R_1} which satisfies

$$\|u_{R_1} - Y_m\|_{L_2(-\pi,\pi)} \mapsto \min \tag{7}$$

is believed to approximate u when $R_1 \to \infty$. Thus the approximation of the sought coefficients S_{mn} is the stationary point of the functional $\|u_{R_1} - Y_m\|_{L_2(-\pi,\pi)}$.

The solution u_{R_1} may be found by any numerical method for elliptic boundary value problems; finite elements technique seems to be promising. Obviously the functional

$\|u_{R_1} - Y_m\|_{L_2(-\pi,\pi)}$ is the second order polynomial with respect to its arguments, so its stationary point is determined by a straightforward way. The convergence of the approach is also controlled by $\|SS^* - I\| \to 0$ as $R_1 \to \infty$. However, dealing with the asymptotics of kind (4) which contains growing and decaying exponents, the special care is necessary to avoid the loss of numerical accuracy due to rounding errors.

As an example of numerics, consider results related to the planar grating with periodic modulation of refraction index χ:

$$\chi(x,z) = \begin{cases} \chi_0 + c\cos x, & 0 < z < d \\ 1, & d < z \end{cases} . \tag{8}$$

The Dirichlet condition at $z = 0$ and the continuity conditions at $z = d$ are assumed for TE-polarized wave.

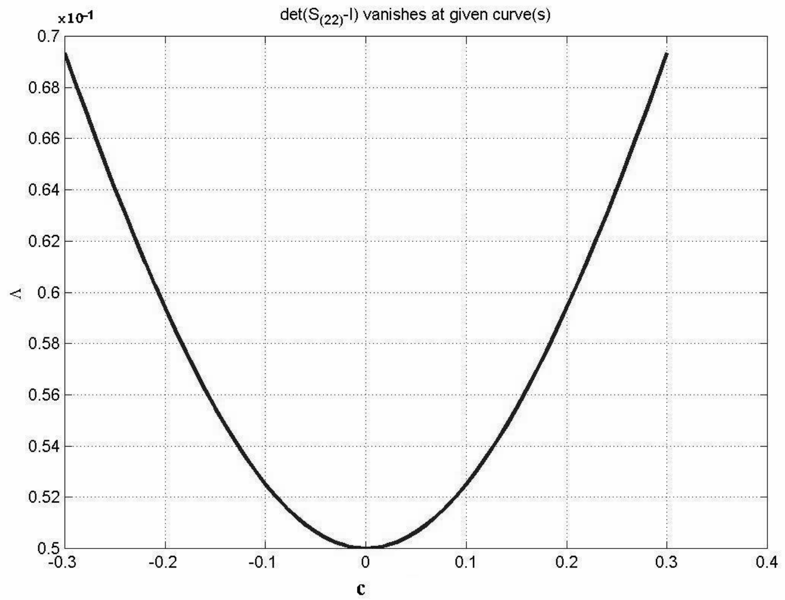

Figure 2: Dependence between frequency and amplitude of the periodic modulation of the refraction index for the zero-order planar grating; $k^2 = \alpha^2 - \Lambda^2$.

Fig. 2 shows the relation between the modulation amplitude c of the refraction index and the frequency under which the surface wave arises in case of the zero-order (that is, $T = 0$) grating. At the extremum point of the curve the parameters k, α, d and χ_0 obey the well known dispersion relation of the planar waveguide.

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